ELSEVIER Journal of Geometry and Physics 26 (1998) 311–328

JOURNAL OF GEOMETRY AND PHYSICS

# Generalized holonomies

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Received 7 May 1997

**Abstract**

We study some issues related to the notion of generalized holonomies, providing a rigorous mathematical framework where we discuss early heuristic ideas from the physics literature, mainly due to R. Gambini and his collaborators, who have tried to formulate an "extended loop representation" of quantum gravity in Ashtekar variables. We also define a BACH (Baker–Campbell–Hausdorff) series for the formal generalized holonomy and prove its convergence in some particular cases. Finally, we discuss the issue of covariance of generalized holonomies, and prove the covariance for nilpotent connections. Copyright © Elsevier Science B.V.

*Subj. Class.:* Quantum gravity; Differential geometry
*1991 MSC:* 55P35, 53C05, 53C45
*Keywords:* Holonomy; Loops; Ashtekar quantum gravity; Baker–Campbell–Hausdorff; Nilpotent connections

## 1. Introduction

Let us begin with the following example. Consider an abelian gauge field theory (source free electromagnetism) on a compact oriented three-dimensional manifold $M$, whose classical (physical) configuration space $\mathcal{C}$ is the space $\Omega^1 M$ of smooth 1-forms, modulo gauge transformations, i.e.,

$$\mathcal{C} \equiv \Omega^1 M/\mathrm{d}C^\infty M.$$

[1] E-mail: jntavar@fc.up.pt. Work partially supported by the research programs: "*Física Matemática*" Praxis/2/2.1/MAT/19/94, and "*Alguns Aspectos da Gravitação Clássica e Quântica*" PBICT/C/MAT /2150/95.

0393-0440/98/$19.00 
*PII* S0393-0440(97)00061-2

As in the scalar field theory an important role in quantum electromagnetism will be played by the dual of the classical configuration space, in a sense the "quantum configuration space":

$$\mathcal{C}^* \equiv (\Omega^1 M/\mathrm{d}C^\infty M)^*.$$

The reason for this is the fact that there are well-defined measures in $\mathcal{C}^*$ and all the cyclic representations of the electromagnetic Weyl algebra can be realized in a Hilbert space:

$$\mathcal{L}^2(\mathcal{C}^*, \mu)$$

consisting of square integrable functions on $\mathcal{C}^*$, with respect to some quasi-invariant measure $\mu$ (see [9]).

Let us now study $\mathcal{C}^*$. This space is the space of DeRham 1-currents $R$, that vanishes on $\mathrm{d}C^\infty M$, i.e., of closed DeRham 1-currents. Recall that we define the *boundary* $\partial R$, of a DeRham 1-current $R$, by $\langle \partial R, f\rangle = \langle R, \mathrm{d}f\rangle\ \forall f \in C^\infty M$, and that $R$ is called closed if $\partial R = 0$.

Every (piecewise smooth) loop $\gamma$ defines an element of $\mathcal{C}^*$, i.e., a DeRham closed 1-current $R_\gamma$ on $M$, by integration

$$R_\gamma(A) = \int_\gamma A, \quad A \in \mathcal{C}.$$

In fact, $R_\gamma(\mathrm{d}f) = 0$.

Let us define the following equivalence relation on the space of piecewise smooth free loops in $M$:

$$\alpha \sim \beta \quad \Leftrightarrow \quad R_\alpha = R_\beta.$$

The quotient space will be denoted by $\mathcal{HL}$, and its elements will be called *holonomic loops*, or briefly *loops* for simplicity. It follows that any (finite) $\mathbb{R}$-linear combination of *loops* belongs to $\mathcal{C}^*$. Let us denote by $\mathcal{HL}_\mathbb{R}$ the $\mathbb{R}$-linear subspace of $\mathcal{C}^*$, generated by all the $R_\gamma$.

**Proposition 1.** *The space $\mathcal{HL}_\mathbb{R}$ is dense in $\mathcal{C}^* \equiv (\Omega^1/\mathrm{d}C^\infty M)^*$ (in the weak $\star$-topology).*

*Proof.* We use the following facts (see [15]):
- (i) The weak $\star$-topology in the dual $X^*$ of a TVS $X$ makes $X^*$ into a locally convex TVS, whose dual $(X^*)^*$ is $X$, i.e., every (weak-$\star$) continuous linear functional on $X^*$ has the form $R \mapsto Rf$ for some $f \in X\ \forall R \in X^*$.
- (ii) As a corollary of Hahn–Banach theorem, in a locally convex TVS $X$, a subspace $S$ is dense iff the only continuous linear functional that vanishes on $S$ is the null functional.

Now if $F$ is a (weak $\star$) continuous linear functional on $X = \mathcal{C}^*$, then, by (i), $F$ takes the form $R \mapsto R\omega$, for some $\omega \in (\Omega^1 M/dC^\infty M)^*$. If $F$ vanishes on $\mathcal{HL}_\mathbb{R}$, then $0 = F(R_\gamma) = R_\gamma(\omega) = f_\gamma \omega\ \forall\gamma$, which implies that $\omega = 0$. So $F = 0$, and by (ii), $\mathcal{HL}_\mathbb{R}$ is dense in $\mathcal{C}^*$. □

We call the elements of the completion $\widetilde{\mathcal{HL}_{\mathbb{R}}}$ of $\mathcal{HL}_{\mathbb{R}}$, (abelian) *generalized loops*, and we will denote them by $\tilde{\alpha}$, $\tilde{\beta}$,.... Therefore, Proposition 1 says that DeRham closed 1-currents are equivalent to generalized loops. Notice that along with the "distributional elements" of the type $R_\alpha$, the space of generalized loops contains also "smooth elements", namely closed 2-forms $e : \omega \to \int_M \omega \wedge e$.

Consider again an abelian gauge field field theory with gauge group $G = U(1)$, so that $\mathcal{G} = \mathrm{i}\mathbb{R}$ and let $A = \mathrm{i}\omega$ be an abelian connection 1-form. In this case we define, for a loop $\gamma \in \mathcal{HL}$, the *holonomy* $U_\gamma(A)$, of $A$ along $\gamma$, by

$$U_\gamma(A) = \mathrm{e}^{\int_\gamma A} = 1 + \sum_{k\geq 1} \frac{(\mathrm{i})^k}{k!} \left( \int_\gamma \omega \right)^k .$$

Note now that we can generalize this definition, the taking instead a loop $\gamma$, a generalized loop $\tilde{\alpha}$ in $\mathcal{HL}_{\mathbb{R}}$, and then define a *generalized holonomy* $\mathbf{U}_{\tilde{\alpha}}(A)$, by

$$\mathbf{U}_{\tilde{\alpha}}(A) = \mathbf{U}_{\tilde{\alpha}}(\mathrm{i}\omega) = \mathrm{e}^{\mathrm{i}\tilde{\alpha}(\omega)} = 1 + \sum_{k\geq 1} \frac{(\mathrm{i})^k}{k!} \tilde{\alpha}(\omega)^k. \tag{1}$$

If $g(x) = \mathrm{e}^{\mathrm{i}f(x)} \in U(1) = \mathrm{i}\mathbb{R}$ is a gauge transformation, then

$$A^g = g^{-1}Ag + g^{-1}\,\mathrm{d}g = A + \mathrm{i}\mathrm{d}f = \mathrm{i}(\omega + \mathrm{d}f)$$

and so

$$\mathbf{U}_{\tilde{\alpha}}(A^g) = \mathrm{e}^{\mathrm{i}\tilde{\alpha}(\omega+\mathrm{d}f)} = \mathrm{e}^{\mathrm{i}\tilde{\alpha}(\omega)}\mathrm{e}^{\mathrm{i}\tilde{\alpha}(\mathrm{d}f)} = \mathrm{e}^{\mathrm{i}\tilde{\alpha}(\omega)} = \mathbf{U}_{\tilde{\alpha}}(A)$$

since $\tilde{\alpha}(\mathrm{d}f) = 0$. So in this case we have gauge covariance (invariance) of the generalized holonomy.

Our aim in this note is to generalize the above concepts in a non-abelian context, considering "non-abelian generalized loops" and non-abelian connection forms. More exactly, we try to give a rigorous mathematical framework where we discuss early heuristic ideas from the physics literature, mainly due to $R$. Gambini and his collaborators, who have tried to formulate an "extended loop representation" of quantum gravity in Ashtekar variables (see [2–4,16]).

The paper is organized as follows. In Section 2, we review the main definitions and properties of generalized loops, based on Chen integrals, as were developed in our early work [18]. In Section 3, we define (formal) generalized holonomies along generalized loops, and study some of its properties. We also define a BACH (Baker–Campbell–Hausdorff) series for the formal generalized holonomy and prove its convergence in some particular cases. Finally, in Section 4, we discuss the issue of covariance of generalized holonomies, recovering the same results of [16], and analyzing the particular case of nilpotent connections.

## 2. The group of generalized loops and its Lie algebra

Let $\mathcal{M}$ be a smooth real compact $n$-dimensional manifold. Let us define the so-called *shuffle algebra* of $\mathcal{M}$. Consider the real vector space $\Omega^1\mathcal{M}$ of real 1-forms on $\mathcal{M}$, and the tensor algebra (over $\mathbb{R}$) of $\Omega^1\mathcal{M}$:

$$\mathcal{T}(\Omega^1\mathcal{M}) = \bigoplus_{r\geq 0}\left(\bigotimes^r \Omega^1\mathcal{M}\right). \tag{2}$$

For simplicity we use the notation

$$\omega_1\cdots\omega_r = \omega_1\otimes\cdots\otimes\omega_r \in \bigotimes^r \Omega^1\mathcal{M}$$

for $r\geq 1$, and set $\omega_1\cdots\omega_r = 1$, when $r=0$. Now we replace the tensor multiplication in $\mathcal{T}(\Omega^1\mathcal{M})$ by the *shuffle multiplication* $\bullet$, defined by

$$\omega_1\cdots\omega_r \bullet \omega_{r+1}\cdots\omega_{r+s} = \sum_\sigma{}' \omega_{\sigma(1)}\cdots\omega_{\sigma(r+s)}, \tag{3}$$

where $\sum'_\sigma$ denotes sum over all $(r,s)$-shuffles, i.e., permutations $\sigma$ of $r+s$ letters with $\sigma^{-1}(1)<\cdots<\sigma^{-1}(r)$ and $\sigma^{-1}(r+1)<\cdots<\sigma^{-1}(r+s)$.

$(\mathcal{T}(\Omega^1\mathcal{M}),\bullet)$ is then an associative, graded commutative real algebra, with unity $1\in\mathbb{R}\subset\mathcal{T}(\Omega^1\mathcal{M})$, which is called the *shuffle algebra* of $\mathcal{M}$ and is denoted by $\mathbf{Sh}(\mathcal{M})$, or simply by $\mathbf{Sh}$. We endow $\mathbf{Sh}(\mathcal{M})$ with the structure of nuclear LMC topological algebra in the way indicated in [18].

**Sh** has also a real Hopf algebra structure. This means (see [1; 17, Chap. XII]) that, in addition to the above real algebra structure, we have three $\mathbb{R}$-linear maps $\Delta:\mathbf{Sh}\to\mathbf{Sh}\otimes\mathbf{Sh}$, called *comultiplication*, $\epsilon:\mathbf{Sh}\to\mathbb{R}$, called *counity*, and $J:\mathbf{Sh}\to\mathbf{Sh}$, called *antipode*, defined, respectively, by the formulas

$$\Delta(\omega_1\cdots\omega_r) = \sum_{i=0}^r \omega_1\cdots\omega_i\otimes\omega_{i+1}\cdots\omega_r, \tag{4}$$

$$\epsilon(\omega_1\cdots\omega_r) = \begin{cases} 0 & \text{if } r\geq 1,\\ 1 & \text{if } r=0,\end{cases} \tag{5}$$

$$J(\omega_1\cdots\omega_r) = (-1)^r\omega_r\cdots\omega_1, \tag{6}$$

which satisfy the usual Hopf algebra identities.

Now, let us fix a point $p\in\mathcal{M}$, and consider the based *loop space* $\mathcal{L}\mathcal{M}_p$ of piecewise smooth loops based at $p$, and the so-called *group of loops* of the manifold $\mathcal{M}$, based at $p$, $(\mathcal{L}\mathcal{M}_p/\sim,\diamond)$, which is denoted by $\mathbf{L}\mathcal{M}_p$. Elements of $\mathbf{L}\mathcal{M}_p$ will be called simply (usual or geometrical) loops, and we denote simply by $\alpha\beta$, the product $\alpha\diamond\beta$ of two elements $\alpha,\beta\in\mathbf{L}\mathcal{M}_p$ (see [18] for definitions and details).

Each loop $\gamma\in\mathbf{L}\mathcal{M}_p$ gives rise to a (continuous) linear functional $X_\gamma$, on $\mathbf{Sh}=\mathbf{Sh}(\mathcal{M})$, defined in each homogeneous element, through *iterated Chen integration*:

$$X_\gamma(\omega_1\cdots\omega_r) = \int_\gamma \omega_1\cdots\omega_r$$
$$= \int_{\Delta_r} f_1(t_1)f_2(t_2)\cdots f_r(t_r)\,\mathrm{d}t_1\,\mathrm{d}t_2\cdots\mathrm{d}t_r, \tag{7}$$

where $\Delta_r = \{(t_1,\ldots,t_r)\in\mathbb{R}^r : 0\le t_1\le\cdots\le t_r\le 1\}$ and $f_j(t) = \omega_j(\gamma(t))\cdot\dot\gamma(t)$.

We deduce, from the properties of the interated Chen integrals, the following properties for these linear functionals $X_\gamma \in \mathbf{Sh}^*$:

$$X_\gamma(\mathbf{u}\bullet\mathbf{v}) = X_\gamma(\mathbf{u})X_\gamma(\mathbf{v})\quad \forall\mathbf{u},\mathbf{v}\in\mathbf{Sh}, \tag{8}$$

i.e., each $X_\gamma$ is a multiplicative linear functional (a character) in $\mathbf{Sh}$, and

$$X_{\alpha\beta} = X_\alpha \star X_\beta \equiv (X_\alpha\otimes X_\beta)\circ\Delta, \tag{9}$$
$$X_{\alpha^{-1}} = X_\alpha\circ J \tag{10}$$

$\forall\alpha,\beta\in\mathbf{L}\mathcal{M}_p$. Moreover, these $X_\gamma$ satisfy the following *differential constraints*:

$$X_\gamma(\mathrm{d}f) = 0, \tag{11}$$
$$X_\gamma((\mathrm{d}f)\omega_1\cdots\omega_r) = X_\gamma((f\omega_1)\omega_2\cdots\omega_r) - f(p)\cdot X_\gamma(\omega_1\cdots\omega_r), \tag{12}$$
$$X_\gamma(\omega_1\cdots\omega_r(\mathrm{d}f)) = (X_\gamma(\omega_1\cdots\omega_2))\cdot f(p) - X_\gamma(\omega_1\cdots\omega_{r-1}(\omega_r f)), \tag{13}$$
$$X_\gamma(\omega_1\cdots\omega_{i-1}(\mathrm{d}f)\omega_{i+1}\cdots\omega_r)$$
$$= X_\gamma(\omega_1\cdots\omega_{i-1}(f\omega_{i+1})\omega_{i+2}\cdots\omega_r) - X_\gamma(\omega_1\cdots(\omega_{i-1}f)\omega_{i+1}\cdots\omega_r), \tag{14}$$

$\forall f\in C^\infty\mathcal{M}$ and for all $\omega_1,\ldots,\omega_r\in\Omega^1\mathcal{M}$.

Let us consider the algebra of functions $\mathcal{A}_p$, defined on the loop group $\mathbf{L}\mathcal{M}_p$, generated by the functions $F^{\omega_1\cdots\omega_r} : \mathbf{L}\mathcal{M}_p \to \mathbf{k}$ defined by

$$F^{\omega_1\cdots\omega_r}(\gamma) = X_\gamma(\omega_1\cdots\omega_r) = \int_\gamma \omega_1\cdots\omega_r. \tag{15}$$

We know that $\mathcal{A}_p$ is a topological LMC algebra of separating functions of $\mathbf{L}\mathcal{M}_p$, which is isomorphic to the quotient algebra $\mathbf{Sh}/\mathbf{J}_p$:

$$\mathbf{Sh}(\mathcal{M})/\mathbf{J}_p \simeq \mathcal{A}_p. \tag{16}$$

Here $\mathbf{J}_p$ is the ideal:

$$\mathbf{J}_p = \mathbf{I}_p + \langle\mathrm{d}C\rangle \tag{17}$$

where $\langle\mathrm{d}C\rangle$ is the ideal generated by $\mathrm{d}C^\infty(\mathcal{M})$ in $\mathbf{Sh}(\mathcal{M})$ and $\mathbf{I}_p$ is the ideal in $\mathbf{Sh}$ generated by all the elements of the type

$$(\mathrm{d}f)\omega_1\cdots\omega_r - (f\omega_1)\omega_2\cdots\omega_r + f(p)\cdot(\omega_1\cdots\omega_r), \tag{18}$$
$$\omega_1\cdots\omega_r(\mathrm{d}f) - (\omega_1\cdots\omega_r)\cdot f(p) + \omega_1\cdots\omega_{r-1}(\omega_r f), \tag{19}$$

and

$$\omega_1 \cdots \omega_{i-1}(\mathrm{d}f)\omega_{i+1} \cdots \omega_r - \omega_1 \cdots \omega_{i-1}(f\omega_{i+1})\omega_{i+2} \cdots \omega_r + \omega_1 \cdots (\omega_{i-1}f)\omega_{i+1} \cdots \omega_r, \tag{20}$$

$\forall f \in C^\infty\mathcal{M}$ and for all $\omega_1, \cdots, \omega_r \in \Omega^1\mathcal{M}$.

The algebra $\mathcal{A}_p$ admits also a real Hopf algebra structure, by defining the comultiplication $\Delta : \mathcal{A}_p \to \mathcal{A}_p \otimes \mathcal{A}_p$, the counity $\epsilon : \mathcal{A}_p \to \mathbf{k}$ and the antipode $J : \mathcal{A}_p \to \mathcal{A}_p$, respectively, by

$$\Delta(F^{\omega_1\cdots\omega_r}) = \sum_{i=0}^{r} F^{\omega_1\cdots\omega_i} \otimes F^{\omega_{i+1}\cdots\omega_r}, \tag{21}$$

$$\epsilon(F^{\omega_1\cdots\omega_r}) = \begin{cases} 0 & \text{if } r \geq 1, \\ 1 & \text{if } r = 0, \end{cases} \tag{22}$$

$$J(F^{\omega_1\cdots\omega_r}) = (-1)^r F^{\omega_r\cdots\omega_1}. \tag{23}$$

Now consider the spectrum $\boldsymbol{\Delta}_p$ of the algebra $\mathcal{A}_p$, consisting of all nonzero continuous characters $\tilde{\alpha} \in \mathcal{A}_p^*$, or equivalently consisting of all nonzero continuous linear functionals $\tilde{\alpha} : \mathbf{Sh} \to \mathbb{R}$ that satisfy the two conditions

$$\tilde{\alpha}(\mathbf{u} \bullet \mathbf{v}) = \tilde{\alpha}(\mathbf{u})\tilde{\alpha}(\mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{Sh}, \tag{24}$$

$$\tilde{\alpha}(\mathbf{J}_p) = 0. \tag{25}$$

Elements of $\boldsymbol{\Delta}_p$ are called *generalized loops*, based at $p \in \mathcal{M}$. We can define a structure of group on $\boldsymbol{\Delta}_p$, through

$$\tilde{\alpha} \star \tilde{\beta} \equiv (\tilde{\alpha} \otimes \tilde{\beta}) \circ \Delta, \tag{26}$$

where we have used the identification $\mathbb{R} \otimes \mathbb{R} \simeq \mathbb{R}$. More explicitly

$$\tilde{\alpha} \star \tilde{\beta}(\omega_1 \cdots \omega_r) = \sum_{i=0}^{r} \tilde{\alpha}(\omega_1 \cdots \omega_i) \cdot \tilde{\beta}(\omega_{i+1} \cdots \omega_r). \tag{27}$$

We also define the inverse of $\tilde{\alpha} \in \boldsymbol{\Delta}_p$, by $\tilde{\alpha} \circ J$, i.e.,

$$\tilde{\alpha}^{-1}(\omega_1 \cdots \omega_r) = (-1)^r \tilde{\alpha}(\omega_r \cdots \omega_1), \tag{28}$$

and take $\epsilon$, given by (5), as the unit element.

We call the above-mentioned topological group $(\boldsymbol{\Delta}_p, .)$, the *group of generalized loops* of $\mathcal{M}$, based at $p \in \mathcal{M}$, and we denote it by $\widetilde{\mathbf{L}\mathcal{M}}_p$.

We have a natural embedding of $\mathbf{L}\mathcal{M}_p$ as a subgroup of $\widetilde{\mathbf{L}\mathcal{M}}_p$, given by the "Dirac map" $X : \mathbf{L}\mathcal{M}_p \to \widetilde{\mathbf{L}\mathcal{M}}_p$, defined by

$$\gamma \mapsto X_\gamma, \tag{29}$$

where $X_\gamma$ is given by (7). Since the functions $F^{\omega_1\cdots\omega_r}$ separate "points" in $\mathbf{L}\mathcal{M}_p$, we see that this is an injective embedding. So we identify $\mathbf{L}\mathcal{M}_p$ with its image under $X$, in $\boldsymbol{\Delta}_p$, and endow $\mathbf{L}\mathcal{M}_p$ with the induced topology. In this topology, a sequence $(\alpha_n)$ converges to $\alpha$ in $\mathbf{L}\mathcal{M}_p$ iff $\lim_{n\to\infty} F^{\mathbf{u}}(\alpha_n) = F^{\mathbf{u}}(\alpha)$, $\forall \mathbf{u} \in \mathbf{Sh}\mathcal{M}$.

Hereafter, we always identify a usual loop $\gamma \in \mathbf{L}\mathcal{M}_p$ with its image $X_\gamma$ in $\widetilde{\mathbf{L}\mathcal{M}}_p \subset \mathbf{Sh}^*$.

We define the *Lie algebra* $\widetilde{l\mathcal{M}}_p$ of the group of generalized loops $\widetilde{\mathbf{L}\mathcal{M}}_p$ as the subspace of $\mathbf{Sh}^*$ consisting of the so-called *point derivations* at $\epsilon$, that vanish on $\mathbf{J}_p$, i.e., an element $\Theta \in \mathbf{Sh}^*$ belongs to $\widetilde{l\mathcal{M}}_p$ iff $\Theta$ satisfies the two conditions

$$\Theta(\mathbf{u} \bullet \mathbf{v}) = \epsilon(\mathbf{u})\Theta(\mathbf{v}) + \Theta(\mathbf{u})\epsilon(\mathbf{v}), \tag{30}$$

$$\Theta(\mathbf{J}_p) = 0. \tag{31}$$

The Lie bracket in $\widetilde{l\mathcal{M}}_p$ is defined through

$$[\Theta_1, \Theta_2] \equiv \Theta_1 \star \Theta_2 - \Theta_2 \star \Theta_1. \tag{32}$$

Note that any point derivation $\Theta$, at $\epsilon$, satisfies

$$\Theta(\omega_1 \cdots \omega_r \bullet \omega_{r+1} \cdots \omega_{r+s}) = 0 \tag{33}$$

$\forall r \geq 1, \forall s \geq 1$, and from this we can deduce that

$$\Theta^n(\omega_1 \cdots \omega_r) = 0 \quad \forall n\ r \geq 0, \tag{34}$$

where $\Theta^{n+1} \equiv \Theta^n * \Theta \forall n \geq 1$.

Now, for each $\Theta \in \widetilde{l\mathcal{M}}_p$, we can define $\exp \Theta$ by

$$\exp \Theta \equiv \epsilon + \sum_{n\geq 1} \frac{\Theta^n}{n!}, \tag{35}$$

where, as always, this means that, for each $\omega_1 \cdots \omega_r$, $\exp \Theta(\omega \cdots \omega_r)$ is defined by

$$\exp \Theta(\omega_1 \cdots \omega_r) = \left(\epsilon + \sum_{n\geq 1} \frac{\Theta^n}{n!}\right)(\omega_1 \cdots \omega_r) \tag{36}$$

if, of course, this series converges. But from (34), it follows that the series (36) is in fact a finite sum, and so $\exp \Theta$ is well defined, in the above sense. Moreover, we can prove that $\exp \Theta$ is a generalized loop, i.e., satisfies conditions (24) and (25).

### 2.1. Example

Let $R : \wedge^1(\mathcal{M}) \to \mathbb{R}$ be a compactly supported closed DeRham 1-current, and define an element $\Theta_R \in \widetilde{l\mathcal{M}}_p$, through

$$\Theta_R(\omega_1 \cdots \omega_r) = \begin{cases} 0 & \text{if } r \neq 1, \\ R(\omega_1) & \text{if } r = 1. \end{cases}$$

Recall that $\Theta_R$ must obey the differential constraints (11) and (12), i.e., $\Theta_R(\mathrm{d}f) = 0$ and $\Theta_R(f\omega) = f(p)\Theta_R(\omega)$. This last condition implies that $\Theta_R$ must be extremely "singular". One such $\Theta_R$ is obtained for $R = \delta_v$, $v \in T_pM$, the Dirac current $\delta_v(\omega) = \omega_p(v)$.

Then we can compute that

$$\begin{aligned}
\exp\Theta_R(\omega_1) &= \Theta_R(\omega_1) = R(\omega_1),\\
\exp\Theta_R(\omega_1\omega_2) &= \frac{1}{2!}R(\omega_1)R(\omega_2),\\
\exp\Theta_R(\omega_1\omega_2\omega_3) &= \frac{1}{3!}R(\omega_1)R(\omega_2)R(\omega_3),\\
\exp\Theta_R(\omega_1\omega_2\omega_3\omega_4) &= \frac{1}{4!}R(\omega_1)R(\omega_2)R(\omega_3)R(\omega_4),
\end{aligned}$$

and so on.

Conversely, given $\tilde{\alpha} \in \mathbf{L}\widetilde{\mathcal{M}}_p$, we define

$$\log\tilde{\alpha} \equiv \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}(\tilde{\alpha}-\epsilon)^n, \tag{37}$$

where $(\tilde{\alpha}-\epsilon)^n \equiv (\tilde{\alpha}-\epsilon)^{n-1}\star(\tilde{\alpha}-\epsilon)$, $\forall n \geq 1$. Since

$$(\tilde{\alpha}-\epsilon)^n(\omega_1\cdots\omega_r) = 0 \quad \forall n > r \geq 0, \tag{38}$$

$\log\tilde{\alpha}$ is also a well-defined element in the above sense, which moreover, belongs to $l\widetilde{\mathcal{M}}_p$.

By the calculus of formal power series, we known that

$$\exp(k\log\tilde{\alpha}) = \tilde{\alpha}^k \quad \forall k \in \mathbb{Z}, \qquad \log(\exp\delta) = \delta.$$

Let us define, for each $t \in \mathbb{R}$:

$$\tilde{\alpha}^t \equiv \exp(t\log\tilde{\alpha}). \tag{39}$$

Then we can easily prove that $t \mapsto \tilde{\alpha}^t$ is a 1-parameter subgroup of $\mathbf{L}\widetilde{\mathcal{M}}_p$, generated by $\log\tilde{\alpha}$, i.e.,

$$\tilde{\alpha}^0 = \epsilon, \quad \tilde{\alpha}^t\star\tilde{\alpha}^s = \tilde{\alpha}^{t+s}, \quad \lim_{t\to 0}\frac{\tilde{\alpha}^t-\epsilon}{t} = \log\tilde{\alpha},$$

this last limit in the above (weak) sense.

## 3. Generalized holonomies

Note that definition (7) works equally well for 1-forms $A$, on $\mathcal{M}$, with values in an associative algebra $\mathcal{A}$ (p.ex., $\mathbb{C}$ or any subalgebra of $gl(p) = gl(p, \mathbb{C})$, the algebra of $p\times p$ complex matrices). Of course, in this case the functions $X_\gamma$, defined by (7), take values on $\mathcal{A}$. So, for example, if $\mathcal{A} \subseteq gl(p)$, then $X_\gamma(A_1A_2) = \int_\gamma A_1A_2$, with $A_1, A_2 \in \Omega^1\mathcal{M}\otimes\mathcal{A}$ i.e., $A_1, A_2$ are two matrices of usual 1-forms in $\mathcal{M}$), denotes the matrix in $\mathcal{A} \subseteq gl(p)$:

$$\left(\int_\gamma A_1A_2\right)^i_j = \int_\gamma (A_1)^i_k\otimes(A_2)^k_j = \int_\gamma (A_1)^i_k(A_2)^k_j \tag{40}$$

and the same for $\int A_1 \cdots A_r$.

$X_\gamma(A_1 \cdots A_2) = \int_\gamma A_1 \cdots A_r$ satisfy the same *differential constraints*, namely (note the order of the products)

$$X_\gamma(\mathrm{d}F) = 0, \tag{41}$$

$$X_\gamma(\mathrm{d}FA_1 \cdots A_r) = X_\gamma((FA_1)A_2 \cdots A_r) - F(p) \cdot X_\gamma(A_1 \cdots A_r), \tag{42}$$

$$X_\gamma(A_1 \cdots A_r \mathrm{d}F) = (X_\gamma(A_1 \cdots A_r)) \cdot F(p) - X_\gamma(A_1 \cdots A_{r-1}(A_r F)), \tag{43}$$

$$\begin{aligned} X_\gamma(A_1 \cdots A_{i-1}(\mathrm{d}F)A_{i+1} \cdots A_r) &= X_\gamma(A_1 \cdots A_{i-1}(FA_{i+1})A_{i+2} \cdot A_r) \\ &\quad - X_\gamma(A_1 \ldots (A_{i-1}F)A_{i+1} \ldots A_r) \end{aligned} \tag{44}$$

$\forall F \in C^\infty \mathcal{M} \otimes \mathcal{A}$ and for all $A_1, \ldots, A_r \in \Omega^1 \mathcal{M} \otimes \mathcal{A}$. (Note, that $A_1 \cdots A_r$ means the product of the matrices $A_1, A_2, \ldots, A_r$, the entries being multiplied through $\otimes$.)

In particular, if $\{T^a\}_{a=1,\ldots,n}$ is a basis for $\mathcal{A}$, and if

$$A = \sum_{a=1}^{n} \omega_a T^a, \quad \omega_a \in \Omega^1(\mathcal{M}), \tag{45}$$

is an $\mathcal{A}$-1-form in $\mathcal{M}$, we can write, using (40):

$$\begin{aligned} \int_\gamma A &= \sum_a \left( \int_\gamma \omega_a \right) T^a, \\ \int_\gamma AA &= \sum_{a,b} \left( \int_\gamma \omega_a \omega_b \right) T^a T^b, \\ \int_\gamma \underbrace{AA \cdots A}_{r} &= \sum_{a_1,\ldots,a_r} \left( \int_\gamma \omega_{a_1} \omega_{a_2} \cdots \omega_{a_r} \right) T^{a_1} T^{a_2} \cdots T^{a_r}. \end{aligned} \tag{46}$$

If $\|A(t)\| = \|A_{\gamma(t)}(\dot\gamma(t))\| \le M \; \forall t \in [0,1]$, Then

$$\begin{aligned} \left\| \int_\gamma \underbrace{AA \cdots A}_{r} \right\| &= \left\| \int_{\Delta_r} A(t_1)A(t_2) \cdots A(T_r)\mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_r \right\| \\ &\le \int_{\Delta_r} \|A(t_1)A(t_2) \cdots A(t_r)\| \mathrm{d}t_1 \mathrm{d}t_2 \cdots \mathrm{d}t_r \\ &\le M^r \, \mathbf{vol}(\Delta_r) = \frac{M^r}{r!} \end{aligned}$$

and so, the series

$$Id + \int_\gamma A + \int_\gamma AA + \int_\gamma AAA + \cdots \tag{47}$$

converges in $Gl(p)$. When $\mathcal{A} = \mathcal{G}$ is the Lie algebra of a Lie group $G \subseteq Gl(p)$, and $A \in \Omega^1(\mathcal{M}) \otimes \mathcal{G}$ represents a connection 1-form, then its *parallel transport (or holonomy)*

$$U : \mathcal{PM} \to G \subseteq Gl(p)$$

is given exactly by the above *chronological series* of interated integrals (see [8] for all these)

$$\begin{aligned} U_\gamma(A) &= Id + \int_\gamma A + \int_\gamma AA + \int_\gamma AAA + \cdots \\ &= Id + \sum_{r>0} \sum_{a_1,\ldots,a_r} \left( \int_\gamma \omega_{a_1}\omega_{a_2}\cdots\omega_{a_r} \right) T^{a_1}T^{a_2}\cdots T^{a_r} \\ &= Id + \sum_{r>0} \sum_{a_1,\ldots,a_r} X_\gamma(\omega_{a_1}\omega_{a_2}\cdots\omega_{a_r}) T^{a_1}T^{a_2}\cdots T^{a_r}. \end{aligned} \tag{48}$$

Under a gauge transformation $g : \mathcal{U} \subseteq \mathcal{M} \to G \subset Gl(p)$, we have that

$$A \to A^g \equiv g^{-1}Ag + g^{-1}\,\mathrm{d}g, \tag{49}$$

and (see [10])

$$U_\gamma(A^g) = g^{-1}(p)U_\gamma(A)g(p), \tag{50}$$

where $p = \gamma(o)$, and so we obtain a gauge independent loop functional, defined by

$$\mathcal{W}_\gamma(A) = \mathbf{Tr}\, U_\gamma(A), \tag{51}$$

which is usually called *Wilson loop variable*.

Now we would like to define *generalized holonomies* and *generalized Wilson loop variables*, through formulas similar to (48) and (51), but instead of the usual loop $\gamma \cong X_\gamma$, we would like to put a generalized loop $\tilde{\alpha} \in \widetilde{\mathbf{L}\mathcal{M}}_p$ (see the discussion in Section 1).

**Definition 1.** Given a connection 1-form $A \in \Omega^1(\mathcal{M}) \otimes \mathcal{G}$, and a generalized loop $\tilde{\alpha} \in \widetilde{\mathbf{L}\mathcal{M}}_p$, we define the *formal generalized holonomy* $U_{\tilde{\alpha}}(A)$, through the formal series

$$\begin{aligned} \mathbf{U}_{\tilde{\alpha}}(A) &\equiv \sum_{r\geq 0} \tilde{\alpha}(\underbrace{AA\cdots A}_{r}) \\ &\equiv Id + \sum_{r>0} \sum_{a_1,\ldots,a_r} \tilde{\alpha}(\omega_{a_1}\omega_{a_2}\cdots\omega_{a_r}) T^{a_1}T^{a_2}\cdots T^{a_r}, \end{aligned} \tag{52}$$

where $\{T^a\}$ is a basis for $\mathcal{G}$, and $A = \sum_a \omega_a T^a$.

Note that the formal generalized holonomy $\mathbf{U}_{\tilde{\alpha}(A)}$, given by (52), is a series in $\mathbb{R}\langle\langle T^a \rangle\rangle$ the algebra of power series in the noncommutative indeterminates $\{T^a\}_{a=1,\ldots,n})$, with coefficients in $\mathbb{R}$.

Every element $\mathbf{F} \in \mathbb{R}\langle\langle T^a \rangle\rangle$ can be written in the form $\mathbf{F} = \sum_{r\geq 0} F_r$, where $F_r$ is a homogeneous form of degree $r$. $\mathbf{F} = \sum_{r\geq 0} F_r \in \mathbb{R}\langle\langle T^a \rangle\rangle$ will be called a *Lie element* if

$F_0 = 0$ and if every $F_r$, with $r > 0$, belongs to the free Lie algebra $\mathcal{L}[T^a]$ (with respect to the bracket $[G, H] = GH - HG$) generated by $\{T^a\}_{(a=1,\ldots,n)}$, over $\mathbb{R}$. Thus note that, in the present context, we are interpreting $\{T^a\}_{a=1,\ldots,n}$ as formal noncommutative indeterminates. By the universal property of free Lie algebras we know that there exists a unique Lie algebra homomorphism

$$\mathcal{L}[T^a] \longrightarrow \mathcal{G}, \tag{53}$$

which sends each formal noncommutative indeterminate $T^a$ in the basis element $T^a$ for $\mathcal{G}$ (we hope that there is no danger of confusion in the use of the same symbol $T^a$ in the previous two contexts).

Recall that given a power series $\mathbf{U} = Id + \mathbf{S} \in \mathbb{R}\langle\langle T^a\rangle\rangle$, we define its *logarithm*, $\mathbf{log\,U} \in \mathbb{R}\langle\langle T^a\rangle\rangle$, through

$$\mathbf{log\,U} = \mathbf{log}(Id + \mathbf{S}) = \sum_{r\geq 1} \frac{(-1)^{r-1}}{r}\mathbf{S}^r. \tag{54}$$

Moreover, for a power series $\mathbf{F} \in \mathbb{R}\langle\langle T^a\rangle\rangle$, with zero constant term, we define its *exponential* by

$$\mathbf{exp\,F} = \sum_{r\geq 0} \frac{\mathbf{F}^r}{r!}. \tag{55}$$

As usual one has the formulas

$$\mathbf{exp}(\mathbf{log}(\mathbf{U})) = \mathbf{U} \quad \text{and} \quad \mathbf{log}(\mathbf{exp}(\mathbf{F})) = \mathbf{F}. \tag{56}$$

Finally, define the symbol $[T^{a_1}, T^{a_2}, \ldots, T^{a_r}]$ inductively by

$$\begin{aligned} [T^{a_1}] &= T^{a_1}, \\ &\vdots \\ [T^{a_1}, T^{a_2}, \ldots, T^{a_r}] &= [[T^{a_1}, T^{a_2}, \ldots, T^{a_{r-1}}], T^{a_r}]. \end{aligned} \tag{57}$$

**Proposition 2.** *If $A \in \Omega^1(\mathcal{M}) \otimes \mathcal{G}$, and $\tilde{\alpha} \in \widetilde{\mathbf{L}\mathcal{M}}_p$, then $\mathbf{F}_{\tilde{\alpha}}(A) \equiv \mathbf{log}(\mathbf{U}_{\tilde{\alpha}(A)})$ is a Lie element. In fact we have that*

$$\begin{aligned} \mathbf{F}_{\tilde{\alpha}}(A) &= \sum_{r>0} (F_{\tilde{\alpha}})_r \\ &= \sum_{r>0} \sum_{a_1,\ldots,a_r} \tfrac{1}{r}(\log\tilde{\alpha})(\omega)a_1\omega_{a_2}\cdots\omega_{a_r})[T^{a_1}, T^{a_2}, \ldots, T^{a_r}], \end{aligned} \tag{58}$$

*where* $\log\tilde{\alpha}$ *was defined in* (37).

*Proof.* That $\mathbf{F}_{\tilde{\alpha}}(A)$ is a Lie element is a direct application of Theorem 3.2 in [14, p. 54], and depends only on the fact that $\tilde{\alpha} : \mathbf{Sh} \to \mathbb{R}$ is an algebra morphism, i.e.,

$$\tilde{\alpha}(\mathbf{u} \bullet \mathbf{v}) = \tilde{\alpha}(\mathbf{u})\tilde{\alpha}(\mathbf{v}) \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{Sh}.$$

So, we see that

$$\mathbf{F}_{\tilde{\alpha}}(A) \equiv \mathbf{log}(\mathbf{U}_{\tilde{\alpha}}(A))$$

can be written in the form

$$\mathbf{F}_{\tilde{\alpha}}(A) = \sum_{r>0} (F_{\tilde{\alpha}})_r,$$

where each $(F_{\tilde{\alpha}})_r$ is homogeneous of degree $r$, and belongs to the free Lie algebra generated by $\{T^a\}_{a=1,\dots,n}$, over $\mathbb{R}$. We can write

$$(F_{\tilde{\alpha}})_r = \sum_{a_1,\cdots,a_r} \Theta(\omega_{a_1}\omega_{a_2}\dots\omega_{a_r})T^{a_1}\cdots T^{a_r}, \tag{59}$$

where

$$\Theta(\omega_1\cdots\omega_k \bullet \omega_{k+1}\cdots\omega_{k+s}) = 0$$

$\forall k \geq 1, \forall_s \geq 1$ (by Theorem 2.2 in [13, p. 214]).

Now substituting

$$\mathbf{S}_{\tilde{\alpha}} \equiv \sum_{r>0}\sum_{a_1,\dots,a_r} \tilde{\alpha}(\omega_{a_1}\omega_{a_2}\cdots\omega_{a_r}\cdots\omega_{a_r})T^{a_1}T^{a_2}\cdots T^{a_r}$$

in (54) and computing, we obtain that

$$\Theta(\omega_{a_1}\omega_{a_2}\cdots\omega_{a_r}) = (\log\tilde{\alpha})(\omega_{a_1}\omega_{a_2}\cdots\omega_{a_r}).$$

Finally, by Dynkin–Specht–Wever theorem (see Theorem 2.3 in [13, p. 214]), we have that

$$r(F_{\tilde{\alpha}})_r = \sum_{a_1,\dots,a_r} (\log\tilde{\alpha})(\omega_{a-1}\omega_{a_2}\cdots\omega_{a_r})[T^{a_1}, T^{a_r}, \dots, T^{a_r}]. \quad \square \tag{60}$$

We call te series $\mathbf{F}_{\tilde{\alpha}}(A)$, given by (58), the BACH (Baker–Campbell–Hausdorff) series for the formal generalized holonomy $\mathbf{U}_{\tilde{\alpha}}(A)$.

When $\tilde{\alpha} = X_\gamma$ is a usual loop, we can given a sufficient condition for the convergence of the corresponding BACH series $\mathbf{F}_{X_\gamma}(A) = \mathbf{F}_\gamma(A)$, using a reasoning similiar to that used in the classical case (see [12]). In fact consider the image in $\mathcal{G}$ of each term $(F_{\tilde{\alpha}})_r$ under the homomorphism (53). Denote it by the same symbol. Consider also a multiplicative norm $\|\cdot\|$ in $\mathcal{G}$, such that $\|[X, Y]\| \leq \|X\|\|Y\|$ (this always exists (see [12])), and let

$$\delta = \max\{\|T^a\|: a = 1, \dots, n\}.$$

Then by induction we have that

$$\|[T^{a_1}, T^{a_2}, \dots, T^{a_r}]\| \leq \delta^r.$$

Now we compute $(\log\tilde{\alpha})(\omega_{a_1}\omega_{a_2}\cdots\omega_{a_r})$. For example, we have

$$\begin{aligned}
\log\tilde{\alpha}(\omega_1) &= \tilde{\alpha}(\omega_1),\\
\log\tilde{\alpha}(\omega_1\omega_2) &= \tilde{\alpha}(\omega_1\omega_2) - \tfrac{1}{2}\tilde{\alpha}(\omega_1)\tilde{\alpha}(\omega_2),\\
\log\tilde{\alpha}(\omega_1\omega_2\omega_3) &= \tilde{\alpha}(\omega_1\omega_2\omega_3) - \tfrac{1}{2}[\tilde{\alpha}(\omega_1)\tilde{\alpha}(\omega_2\omega_3) + \tilde{\alpha}(\omega_1\omega_2)\tilde{\alpha}(\omega_3)]\\
&\quad + \tfrac{1}{3}\tilde{\alpha}(\omega_1)\tilde{\alpha}(\omega_2)\tilde{\alpha}(\omega_3),\\
\log\tilde{\alpha}(\omega_1\omega_2\omega_3\omega_4) &= \tilde{\alpha}(\omega_1\omega_2\omega_3\omega_4) - \tfrac{1}{2}[\tilde{\alpha}(\omega_1)\tilde{\alpha}(\omega_2\omega_3\omega_4) + \tilde{\alpha}(\omega_1\omega_2)\tilde{\alpha}(\omega_3\omega_4)\\
&\quad + \tilde{\alpha}(\omega_1\omega_2\omega_3)\tilde{\alpha}(\omega_4)] + \tfrac{1}{3}[\tilde{\alpha}(\omega_1)\tilde{\alpha}(\omega_2)\tilde{\alpha}(\omega_3\omega_4)\\
&\quad + \tilde{\alpha}(\omega_1)\tilde{\alpha}(w_2\omega_3)\tilde{\alpha}(\omega_4) + \tilde{\alpha}(\omega_1\omega_2)\tilde{\alpha}(\omega_3)\tilde{\alpha}(\omega_4)]\\
&\quad - \tfrac{1}{4}\tilde{\alpha}(\omega_1)\tilde{\alpha}(\omega_2)\tilde{\alpha}(\omega_3)\tilde{\alpha}(\omega_4),
\end{aligned}$$

and so on. Now with $\tilde{\alpha} = X_\gamma$ each term is given by Chen interated integration, and we have that

$$|X_\gamma(\omega_{a_1}\omega_{a_2}\cdots\omega_{a_r})| \le \frac{M^r}{r!}, \tag{61}$$

where

$$M = \max\{|X_\gamma(w_a)|: a = 1,\ldots,n\}.$$

So we obtain

$$\begin{aligned}
\|(F_\gamma)_r\| &\le \left\| \frac{1}{r}\sum_{a_1,\ldots,a_r} (\log\tilde{\alpha})(\omega_{a_1}\omega_{a_2}\cdots\omega_{a_r})[T^{a_1}T^{a_2},\ldots,T^{a_r}] \right\|\\
&\le \sum_{k=1}^{r} D_r\Lambda^r,
\end{aligned} \tag{62}$$

with (recall that $n = \dim\mathcal{G}$):

$$\Lambda = nM\delta,$$

and

$$D_r = \frac{1}{r}\sum_{k=1}^{r}\frac{1}{k}\sum_{j_1,\ldots,j_k}\frac{1}{j_1!,\ldots j_k!},$$

where the sum $\sum_{j_1,\ldots,j_k}$ is made for all $j_1 \ge 1,\ldots,j_k \ge 1$ such that $j_1 + \cdots + j_k = r$. Now the term $\sum_{j_1,\ldots,j_k} 1/j_1!\cdots j_k!$ is the coefficient in $t^r$ of the Taylor series in $t = 0$ of $(\mathrm{e}^t - 1)^k$, and so $rD_r$ is the coefficient in $t^r$ of the Taylor series in $t = 0$ of

$$\sum_{k=1}^{r}\frac{1}{k}(\mathrm{e}^t - 1)^k,$$

or, what is the same, of

$$f(t) = \sum_{k\ge 1}\frac{1}{k}(\mathrm{e}^t - 1)^k.$$

We compute that

$$\sum_{r\geq 1} D_r \Lambda^r = \int_0^{\Lambda} \frac{f(t)}{t}\, \mathrm{d}t.$$

But the series for $f(t)$ converges $\forall t\colon |\mathrm{e}^t - 1| < 1$, i.e., $\forall t\colon t < \log 2$, and so $\sum_{k\geq 1} D_r \Lambda^r$ converges if $\Lambda < \log 2$. Thus, by (62), we see that the BACH series $\mathbf{F}_\gamma(A)$ converges if

$$\Lambda = nM\delta < \log 2. \tag{63}$$

*3.1. Examples*

(i) When the connection is abelian, say $A = \mathrm{i}\omega$, then

$$\mathbf{U}_{\tilde{\alpha}}(\mathrm{i}\omega) = 1 + \sum_{k\geq 1} \frac{\mathrm{i}^k}{k!} \tilde{\alpha}(\omega)^k,$$

and we recover formula (1) of Section 1. The corresponding BACH formula is

$$\mathbf{F}_{\tilde{\alpha}}(\mathrm{i}\omega) = \mathrm{i} \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \tilde{\alpha}(\omega)^k,$$

and so is convergent if $|\tilde{\alpha}(\omega)| < 1$.

(i) if $\tilde{\alpha} = \exp \Theta_R$ like in the Example of Section 2.1, then

$$\mathbf{F}_{\tilde{\alpha}}(A) = \sum_a R(\omega_a) T^a \in \mathcal{G}.$$

(ii) If $\tilde{\alpha} = X_\gamma^t \equiv \exp(t \log X_\gamma)$, then

$$\mathbf{F}_{X_\gamma^t}(A) = t\mathbf{F}_{X_\gamma}(A) \in \mathcal{G},$$

if condition (63) is verified.

**Proposition 3.** *Let $A \in \Omega^1(\mathcal{M}) \otimes \mathcal{G}$. Then the set $\mathbf{G} \equiv \{\mathbf{U}_{\tilde{\alpha}}(A)\}_{\tilde{\alpha}}$ of formal generalized holonomies, it is a group. In fact*

$$\mathbf{U}_{\tilde{\alpha}}(A)\mathbf{U}_{\tilde{\beta}}(A) = \mathbf{U}_{\tilde{\alpha}\star\tilde{\beta}}(A), \qquad [U_{\tilde{\alpha}}(A)]^{-1} = \mathbf{U}_{\tilde{\alpha}^{-1}}(A),$$

*$\forall \tilde{\alpha}, \tilde{\beta} \in \widetilde{\mathbf{L}\mathcal{M}}_p$, where*

$$[\mathbf{U}_{\tilde{\alpha}}(A)]^{-1} = Id + \sum_{r>0} \sum_{a_1\cdots a_r} (-1)^r (\tilde{\alpha}(\omega_{a_r}\omega_{a_{r-1}} \cdots \omega_{a_1})) T^{a_1} T^{a_2} \cdots T^{a_r}.$$

*So the map $\tilde{\alpha} \mapsto \mathbf{U}_{\tilde{\alpha}}(A)$ is a homomorphism of groups $\widetilde{\mathbf{L}\mathcal{M}}_p \to \mathbf{G}$.*

*Proof* (see also Corollary 3.3 in [14, p. 55]).

$$
\begin{aligned}
U_{\tilde\alpha}(A)U_{\tilde\beta}(A) &= (Id + \tilde\alpha(\omega_{a_1})T^{a_1} + \cdots) + \tilde\alpha(\omega_{a_1}\omega_{a_2})T^{a_1}T^{a_2} + \cdots) \\
&\quad \times (Id + \tilde\beta(\omega_{a_1})T^{a_1} + \cdots) + \tilde\beta(\omega_{a_1}\omega_{a_2})T^{a_1}T^{a_2} + \cdots) \\
&= Id + (\tilde\alpha(\omega_{a_1}) + \tilde\beta(\omega_{a_1}))T^{a_1} \\
&\quad + (\tilde\alpha(\omega_{a_1}\omega_{a_2}) + \tilde\alpha(\omega_{a_1})\tilde\beta(\omega_{a_2}) + \tilde\beta(\omega_{a_1}\omega_{a_2}))T^{a_1}T^{a_2} \\
&\quad + \cdots + (\tilde\alpha(\omega_{a_1}\cdots\omega_{a_r}) + \tilde\alpha(\omega_{a_1})\tilde\beta(\omega_{a_2}\cdots\omega_{a_r}) \\
&\quad + \cdots + \tilde\beta(\omega_{a_1}\cdots\omega_{a_r}))T^{a_1}\cdots T^{a_r} + \cdots \\
&= Id + (\tilde\alpha \star \tilde\beta)(\omega_{a_1})T^{a_1} + (\tilde\alpha \star \tilde\beta)(\omega_{a_1}\omega_{a_2})T^{a_1}T^{a_2} \\
&\quad + \cdots + (\tilde\alpha \star \tilde\beta)(\omega_{a_1}\cdots\omega_{a_r})T^{a_1}\cdots T^{a_r} + \cdots \\
&= U_{\tilde\alpha\star\tilde\beta}(A). \qquad \square
\end{aligned}
$$

## 4. (Non)covariance of generalized holonomies

Now let $g : \mathcal{M} \to G \subset Gl(p)$ be a gauge transformation and $A \in \Omega^1(\mathcal{M}) \otimes \mathcal{G}$ a connection 1-form. $g$ acts on $A$ by

$$A \mapsto A^g = g^{-1}Ag + g^{-1}\,\mathrm{d}g.$$

To obtain the corresponding infinitesimal action put $g(t) = \mathrm{e}^{t\xi}$, so that $g(o) = Id$ and

$$\xi = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} g(t) : \mathcal{M} \to \mathcal{G} \subset gl(p)$$

is an *infinitesimal gauge transformation*. Then the infinitesimal affine action on $A$ is given by

$$\xi \mapsto A^\xi = A + D_A\xi, \tag{64}$$

where $D_A\xi = \mathrm{d}\xi + A\xi - \xi A = \mathrm{d}\xi + [A, \xi]$ is the covariant derivative of $\xi$, i.e., $D_A\xi$ is a tangent vector in $A$ to the affine space of gauge connection 1-forms.

Now let $\tilde\alpha \in \widetilde{\mathbf{L}\mathcal{M}}_p$. We want to study the change in the formal generalized holonomy when the connection $A$ suffers an infinitesimal gauge transformation $A \mapsto A^\xi$. So we want to compute $\mathbf{U}_{\tilde\alpha}(A^\xi)$, using only the differential constraints (41)–(44). However, to simplify matters we compute the "differential" of $\mathbf{U}_{\tilde\alpha}$ at $A$:

$$\mathrm{d}(\mathbf{U}_{\tilde\alpha})_A(D_A\xi) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{U}_{\tilde\alpha}(A + tD_A\xi). \tag{65}$$

Calling $B = D_A\xi = \mathrm{d}\xi + [A, \xi]$ we have formally the following:

$$
\begin{aligned}
\mathrm{d}(\mathbf{U}_{\tilde\alpha})_A(D_A\xi) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{U}_{\tilde\alpha}(A + tD_A\xi) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathbf{U}_{\tilde\alpha}(A + tB)
\end{aligned}
$$

$$
\begin{aligned}
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \sum_{r\geq 0} \tilde{\alpha}((A+tB)^r) \\
&= \tilde{\alpha}(B) + \tilde{\alpha}(AB+BA) + \tilde{\alpha}(AAB+ABA+BAA) + \cdots
\end{aligned}
$$

Now using the differential constraints (41)–(44), and denoting $C = [A, \xi]$, we have

$$
\begin{aligned}
&\tilde{\alpha}(B) = \tilde{\alpha}(\mathrm{d}\xi + C) = \tilde{\alpha}(C), \\
&\tilde{\alpha}(AB+BA) = [\tilde{\alpha}(A), \xi(p)] - \tilde{\alpha}(C) + \tilde{\alpha}(AC+CA), \\
&\tilde{\alpha}(AAB+ABA+BAA) \\
&\quad = [\tilde{\alpha}(AA), \xi(p)] - \tilde{\alpha}(AC+CA) + \tilde{\alpha}(AAC+ACA+CAA), \\
&\tilde{\alpha}(AAAB+AABA+ABAA+BAAA) \\
&\quad = [\tilde{\alpha}(AAA), \xi(p)] - \tilde{\alpha}(AAC+ACA+CAA) \\
&\qquad + \tilde{\alpha}(AAAC+AACA+ACAA+CAAA),
\end{aligned}
$$

and so on, and so formally

$$
\mathrm{d}(\mathbf{U}_{\tilde{\alpha}})_A(D_A\xi) = \sum_{n\geq 1}([\tilde{\alpha}(\underbrace{AA\cdots A}_{n-1}), \xi(p)] + R_n(\tilde{\alpha}; A, \xi) - R_{n-1}(\tilde{\alpha}; A, \xi)), \tag{66}
$$

where

$$
\begin{aligned}
R_n(\tilde{\alpha}; A, \xi) &= \tilde{\alpha}\left(\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} (A + s[A,\xi])^n\right) \\
&= \tilde{\alpha}(\underbrace{AA\cdots A}_{n-1}[A,\xi] + \underbrace{AA\cdots A}_{n-2}[A,\xi]A + \cdots + [A,\xi]\underbrace{AA\cdots A}_{n-1}).
\end{aligned} \tag{67}
$$

Consider the partial sum of the $N \geq 1$ first terms of the series (66):

$$
S_N \equiv \left[\sum_{n=1}^{N} \tilde{\alpha}(\underbrace{AA\ldots A}_{n-1}), \xi(p)\right] + R_N(\tilde{\alpha}; A, \xi) \tag{68}
$$

(we put $\tilde{\alpha}(AA\cdots A)$ ($(n-1)$ times) $= Id$ for $n = 1$). So we see that if $\sum_{n=1}^{N} \tilde{\alpha}(AA\cdots A)$ converges to $\mathbf{U}_{\tilde{\alpha}}(A)$ (in $G \subset GL(p)$, when $N \to \infty$), then the formal generalized holonomy $\mathbf{U}_{\tilde{\alpha}}$ will be gauge covariant iff:

$$
\lim_{N\to\infty} R_N(\tilde{\alpha}; A, \xi) = 0. \tag{69}
$$

Thus we obtain the same result of Schilling [16], who has also given several examples of noncovariance of generalized holonomies.

However, if we work with nilpotent connections, i.e., those for which $A^r = 0$ for some $r \geq 1$, then everything works well. In fact, assume that $A \in \Omega^1\mathcal{M} \otimes \mathcal{N}_r$, where $\mathcal{N}_r$ denotes the Lie algebra of nilpotent upper triangular $(r+1)\times(r+1)$ matrices. In this case, the series (52) for $\mathbf{U}_{\tilde{\alpha}}$ is finite and so convergent. For example, if $\omega_1,\ldots,\omega_r\in\Omega^1\mathcal{M}$ and

$$A=\begin{bmatrix} 0 & \omega_1 & 0 & \cdots & \cdots & 0\\ 0 & 0 & \omega_2 & 0 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 0 & \omega_r\\ 0 & 0 & 0 & \cdots & 0 & 0\end{bmatrix}$$

then, for every generalized loop $\tilde{\alpha}$, we have

$$\mathbf{U}_{\tilde{\alpha}}(A)=\begin{bmatrix} 1 & \tilde{\alpha}(\omega_1) & \tilde{\alpha}(\omega_1\omega_2) & \tilde{\alpha}(\omega_1\omega_2\omega_3) & \cdots & \tilde{\alpha}(\omega_1\omega_2\cdots\omega_r)\\ 0 & 1 & \tilde{\alpha}(\omega_2) & \tilde{\alpha}(\omega_2\omega_3) & \cdots & \tilde{\alpha}(\omega_2\cdots\omega_r)\\ 0 & 0 & 1 & \tilde{\alpha}(\omega_3) & \cdots & \tilde{\alpha}(\omega_3\cdots\omega_r)\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 0 & 0 & 0 & \cdots & 1 & \tilde{\alpha}(\omega_r)\\ 0 & 0 & 0 & \cdots & 0 & 1\end{bmatrix}.$$

Moreover in this case condition (69) is verified, and so $\mathbf{U}_{\tilde{\alpha}}(A)$ is covariant for every generalized loop $\tilde{\alpha}$.

However, in the general case this seems not to be true, first because it seems very difficult to give a general criterion for convergence of the series (52) for $\mathbf{U}_{\tilde{\alpha}}$, and second because condition (69) is not always verified, even if the series (52) converges! (see the examples in [16]).

## Acknowledgements

I am extremely grateful to J. Mourão for many helpful comments and criticisms which substantially improved the presentation. I also thank the useful comments of C.M. Menezes, and the members of the TQFT club, namely A. Caetano, M. Mackaay, N. Manojlovic, R. Picken, P. Semiao and J. Velhinho.

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